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On the Application of Buchberger's Algorithm to Automated Geometry Theorem Proving

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In this paper we present a new approach to automated geometry theorem proving that is based on Buchberger's Gröbner bases method. The goal is to automatically prove geometry theorems whose hypotheses and conjecture can be expressed algebraically, i.e. by polynomial equations. After shortly reviewing the problem considered and discussing some new aspects of confirming theorems, we present two different methods for applying Buchberger's algorithm to geometry theorem proving, each of them being more efficient than the other on a certain class of problems. The second method requires a new notion of reduction, which we call pseudoreduction. This pseudoreduction yields results on polynomials over some rational function field by computations that are done merely over the rationals and, therefore, is of general interest also. Finally, computing time statistics on 70 non-trivial examples are given, based on an implementation of the methods in the computer algebra system SAC-2 on an IBM 4341.

1. Introduction

In 1948, A. Tarski described a decision method for the theory of elementary algebra and geometry. Unfortunately, its complexity was found to be so great that it was of no practical value. In 1978, Wu Wen-Tsün described an algorithm that succeeds in many cases in mechanically proving "insufficiently specified" geometrical theorems that can be formulated in an algebraic way involving only polynomial equations (Wu, 1978). The method, which is explained in detail in Wu (1984*a*), is based on the work done by J. F. Ritt. Since this approach involves complex numbers, whereas geometry theorems are assertions about real numbers, only confirmations of geometry theorems can be achieved.

In Kutzler & Stifter (1986*a*) we reported on two approaches how to apply the method of Gröbner bases to mechanical geometry theorem proving. Gröbner bases have been introduced in Buchberger (1965, 1970) together with an algorithm for computing such bases. Assuming some familiarity with the Gröbner bases method (for a tutorial, see Buchberger (1985)), we summarise our results without giving proofs of the lemmas cited. Actually, all proofs can be found in Kutzler & Stifter (1986*a*). We will use Buchberger's algorithm for constructing a reduced Gröbner basis from a given ideal basis F ("Gröbner basis of F") and a normal form algorithm that computes a normal form of a polynomial p modulo a given ideal basis F ("normal form of p modulo F"). Only confirmations of geometry theorems are possible.

2. Algebraic Formulations of Geometry Theorems

We start from algebraic formulations of geometry theorems, i.e. from formulations in which geometrical entities are described by relations between Cartesian coordinates. The

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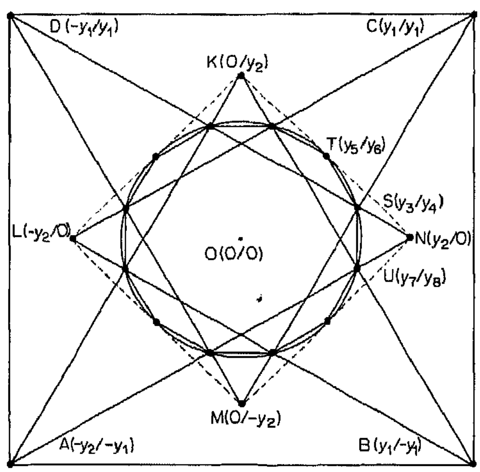
methods presented here are restricted to theorems whose premises and conclusion(s), roughly, are polynomial equations. The intellectual process involved in the translation of geometry theorems into algebraic ones and the relevance of the particular class of equational theorems has been extensively discussed in the papers by Wu Wen-Tsün (see, for example, Wu, 1984*a*) and subsequent authors. The premises are called hypotheses there. Roughly, in many cases one can find an adequate algebraic formulation of a given geometry problem by choosing some coordinate system and coordinates for the points of the geometrical entities involved and expressing the hypotheses and the conjecture in form of polynomial equations using elementary analytic geometry.

EXAMPLE. (International Mathematical Olympiad, 1977, see Fig. 1): Let ABCD be a square and let K, L, M, N be such that ABK, BCL, CDM and DAN are regular triangles, either all of them interior or all of them exterior to the square. Then the midpoints of KL, LM, MN, NK and the intersection points CL $\cap$ BK, CM $\cap$ DN, BK $\cap$ AN, BL $\cap$ CM, AN $\cap$ DM, AK $\cap$ BL, DM $\cap$ CL, DN $\cap$ AK form a regular 12-gon.



A possible algebraic formulation of this geometry problem is:

hypotheses:

$$h_1 = y_1^2+(y_2+y_1)^2-4y_1^2 = 0 \qquad (\Delta\text{ABK is regular}),$$
$$h_2 = y_3(y_1+y_2)-y_1(y_4+y_2) = 0,$$
$$h_3 = y_1(y_3-y_2)+y_4(y_2+y_1) = 0 \quad (\text{S} = \text{CM} \cap \text{DN}),$$
$$h_4 = 2y_5-y_2 = 0,$$
$$h_5 = 2y_6-y_2 = 0 \qquad (\text{T is midpoint of KN}),$$
$$h_6 = y_7(y_1+y_2)+y_1(y_8-y_2) = 0,$$
$$h_7 = y_1(y_7-y_2)-y_8(y_2+y_1) = 0 \quad (\text{U} = \text{BK} \cap \text{AN}),$$

conjectures:

$$c_1 = (y_5-y_3)^2+(y_6-y_4)^2-(y_3-y_7)^2-(y_4-y_8)^2 = 0 \quad (\text{length ST} = \text{length SU}),$$
$$c_2 = y_3^2+y_4^2-y_5^2-y_6^2 = 0 \qquad (\text{length OS} = \text{length OT}).$$

In general, the hypotheses and the conjecture are not sufficient to prove theorems. For most theorems one also has to exclude some cases (for instance "degenerate cases") by means of an inequality $s \neq 0$. (An exact analysis of several examples shows that in many cases the polynomial s depends on the algebraic translation rather than on the geometry problem itself.) Therefore, following Wu (1984*a*), the problem is

PROBLEM P1.

Given: $H = \{h_1, \ldots, h_n\} \subset \mathbb{Q}[y_1, \ldots, y_m]$, $\quad c \in \mathbb{Q}[y_1, \ldots, y_m]$.

Find: $s \in \mathbb{Q}[y_1, \ldots, y_m]$, such that

(1) $(\exists\, z_1, \ldots, z_m)(H(z_1, \ldots, z_m) = 0 \wedge s(z_1, \ldots, z_m) \neq 0) \wedge$

(2) $(\forall\, z_1, \ldots, z_m)(H(z_1, \ldots, z_m) = 0 \wedge s(z_1, \ldots, z_m) \neq 0 \Rightarrow c(z_1, \ldots, z_m) = 0)$.

(If no such s exists, report this fact.)

(Here and in the sequel $H(z_1, \ldots, z_m) = 0$ is an abbreviation for

$$h_1(z_1, \ldots, z_m) = 0 \wedge \ldots \wedge h_n(z_1, \ldots, z_m) = 0.$$

In case there is more than one conjecture (which should be proved independently) one can regard several problems of type P1, one for each of the conjectures.)

Wu gave a complete decision procedure for this problem in algebraically closed fields using Ritt's bases (Wu, 1984*a*). His algorithm requires decomposition of algebraic varieties into irreducible components. Up to today no full implementation exists. The implemented version, called the "China-Prover" (Wu, 1984*b*), can only confirm problems of the above type over complex variables. An implementation of Wu's method involving factorisation of quadratic polynomials is reported in Chou (1985).

However, complete procedures in algebraically closed fields are of theoretical interest only, since they involve complex numbers, whereas geometry problems are assertions about real numbers. Refutations of geometry theorems are not possible using this approach. In general, not even confirmations of geometry theorems, i.e. verifications of (1) and (2) for real variables can be derived automatically from a proof in an algebraically closed field. If one finds a polynomial s such that (1) and (2) hold for complex variables, (2) is fulfilled for the variables ranging over $\mathbb{R}$, but not necessarily (1).

3. Confirmations of Geometry Theorems

The methods we describe in the next sections allow to confirm geometry theorems on the basis of an assumption about the independence of some of the variables. This assumption can be motivated by considering a possible "construction process" of the geometrical entities where "independent" variables are used for describing points that can be chosen arbitrarily, "dependent" variables are used for describing points that have to meet certain properties in relation to the points introduced earlier in the "construction". This coincides with the following exact notion of independence (see, for instance, Gröbner, 1949).

DEFINITION. $y_{i1}, \ldots, y_{ir}$ $(1 \leqslant i1 \leqslant \ldots \leqslant ir \leqslant m)$ are *independent* w.r.t. Ideal(H) iff

$$\mathrm{Ideal}(H) \cap \mathbb{Q}[y_{i1}, \ldots, y_{ir}] = \{0\}.$$

In the sequel, let $u=(y_{i1},\ldots,y_{ir})$ and let

$$x=(y_1,\ldots,y_{i1-1},y_{i1+1},\ldots,y_{ir-1},y_{ir+1},\ldots,y_m).$$

We restrict ourselves to subsidiary conditions in the independent variables. This means that only restrictions on the points chosen arbitrarily are considered. Actually, we consider the following:

PROBLEM P2.

Given: $H\subset \mathbf{Q}[u,x]$, $c\in\mathbf{Q}[u,x]$, u independent w.r.t. Ideal(H).

Find: $s\in\mathbf{Q}[u]\backslash\{0\}$ such that $s.c\in\text{Ideal}(H)$.
(If no such s exists, report this fact.)

Such an s satisfies conditions (1) and (2) of P1, as a straightforward computation shows. Unfortunately, the reasoning is over the complex numbers. As stated above, (2) carries over to the reals. So it remains to ensure (1) for real variables, i.e. it remains to show $(\exists\, u,x\in\mathbb{R})\ (H(u,x)=0 \wedge s(u)\neq 0)$. This is certainly true if the geometrical object considered exists in "sufficiently many" instances, i.e. if there is a non-void open set $0\subset\mathbb{R}^r$ such that, for all $u\in 0$, $H(u,x)=0$ for some $x\in\mathbb{R}^{m-r}$. For, $s(u)=0$ defines a set of dimension $r-1$ at most, so points $u\in 0$ remain where $s(u)\neq 0$.

Often, an algebraic formulation H of geometrical facts is ambiguous for degenerate instances of the geometrical object considered. This requires the exclusion of these cases by means of the polynomial s. (Actually, the interpretation of a polynomial inequality $s\neq 0$ in terms of geometrical properties, i.e. the retranslation of an algebraic subsidiary condition into a geometrical condition, is sometimes difficult.) From a geometrical point of view it is reasonable to abolish this ambiguity by restricting attention to the instances neighbouring non-degenerate instances of the entities. This goal is readily achieved by considering only the "proper zeros" of the hypothesis polynomials, i.e. those points where the dependent variables depend continuously on the independent variables.

DEFINITION. *(u,x) is a proper zero of H* iff

$$H(u,x)=0 \wedge (\forall\,\varepsilon>0)(\exists\,\delta>0)(\forall\,u':|u'-u|<\delta)(\exists\,x':|x'-x|<\varepsilon)(H(u',x')=0).$$

The following theorem ensures that at proper points one needs no subsidiary conditions.

THEOREM. *Let $H\subset\mathbf{Q}[u,x]$ and $c\in\mathbf{Q}[u,x]$. Then*

$$(\exists\, s\in\mathbf{Q}[u]\backslash\{0\})(s.c\in\text{Ideal}_{\mathbf{Q}[u,x]}(H))\Rightarrow(\forall\,u,x)((u,x) \textit{ is a proper zero of } H\Rightarrow c(u,x)=0).$$

In the sequel we describe two methods based on Buchberger's algorithm which decide whether the variables u are independent w.r.t. Ideal (H) and solve problem P2.

4. Two Methods Based on Buchberger's Algorithm

The first method we present makes use of the fact that problem P2 is equivalent to the "Main Problem of Ideal Theory" for the polynomial ring over the field of the rational functions in the independent variables, which is readily solved by a straightforward application of Buchberger's algorithm.

ALGORITHM. RED (in: H, c, u)
G: = Gröbner basis of H in $\mathbf{Q}(u)[x]$
if $G = \{1\}$ *then output* "variables not independent"; *stop*
r: = normal form of c modulo G in $\mathbf{Q}(u)[x]$
if $r = 0$ *then output* "Theorem confirmed"
else output "Theorem not confirmed".

The correctness of RED follows from the following lemmas, the termination follows from the termination of Buchberger's algorithm and the normal form algorithm. Proofs can be found in Kutzler & Stifter (1986*a*).

LEMMA. *Let* $H \subset \mathbf{Q}[u, x]$ *and* $c \in \mathbf{Q}[u, x]$. *Then*

$$(\exists\, s \in \mathbf{Q}[u]\backslash\{0\})(s \,.\, c \in Ideal_{\mathbf{Q}[u,x]}(H)) \text{ iff } c \in Ideal_{\mathbf{Q}(u)[x]}(H).$$

LEMMA. *The variables u are independent w.r.t.* $Ideal_{\mathbf{Q}[u,x]}(H)$ *iff* $1 \notin Ideal_{\mathbf{Q}(u)[x]}(H)$.

EXAMPLE. For the above example, the Gröbner basis of the hypotheses over the rational functions in y_1 is:

$$g_1 = y_2^2+2y_1y_2-2y_1^2, \quad g_2 = y_3-\tfrac{1}{2}y_1, \quad g_3 = y_4+\tfrac{1}{2}y_2-\tfrac{1}{2}y_1,$$
$$g_4 = y_5-\tfrac{1}{2}y_2, \quad g_5 = y_6-\tfrac{1}{2}y_2, \quad g_6 = y_7-\tfrac{1}{2}y_1,$$
$$g_7 = y_8-\tfrac{1}{2}y_2+\tfrac{1}{2}y_1.$$

$c_1 \in \text{Ideal}(G)$, $c_2 \in \text{Ideal}(G)$ and hence the theorem is confirmed.

Our second approach is equivalent to RED but shows quite a different efficiency. (None of them is faster in general, but each one runs faster on a certain class of problems.) The method is based on a new notion of reduction, called pseudoreduction. It uses computation in the polynomial ring over the rationals. Our notion of "pseudoreduction" relates to reduction in the same way as the well-known pseudodivision relates to division.

DEFINITION: Let $r, p, g \in \mathbf{Q}[u, x]$. r *u-pseudoreduces* to p modulo g iff $lc_{\mathbf{Q}(u)[x]}(g) \,.\, r \rightarrow_g^* p$ and p is in normal form modulo g. (Here $lc_{\mathbf{Q}(u)[x]}(g)$ denotes the leading coefficient of g regarded as a polynomial in $\mathbf{Q}(u)[x]$.) For abbreviation we write $r \underset{u\,g}{\curvearrowright} p$.

In the following we use a purely lexicographical ordering on $\mathbf{Q}[u, x]$ with $u < x$, denoted by $<$, and a purely lexicographical ordering on $\mathbf{Q}(u)[x]$, such that the variables x are ordered like in the order $<$, denoted by $<_u$. The following algorithm also decides the independence of the variables u w.r.t. Ideal(H) and solves P2.

ALGORITHM. PRED (in: H, c, u)
G: = Gröbner basis of H in $\mathbf{Q}[u, x]$
if $(\exists\, g \in G)(g \in \mathbf{Q}[u])$ *then output* "variables not independent"; *stop*
(r, s): = ITPSRED(G, c, u)
if $r = 0$ *then output* "Theorem confirmed"
else output "Theorem not confirmed".

ALGORITHM. ITPSRED (in: G, c, u out: r, s)
$r := $ normal form of c modulo G in $\mathbf{Q}[u, x]$; $s := 1$
while exist $g \in G$, r' such that $r_{u_g} r'$ *do*
  Choose g, r' such that $r_{u_g} r'$
  $s := s \cdot lc_{\mathbf{Q}(u)[x]}(g)$
  $r := $ normal form of r' modulo G in $\mathbf{Q}[u, x]$
enddo

The correctness of PRED follows from the following lemma and theorem. Their proofs as well as the termination proof can be found in Kutzler & Stifter (1986*a*).

LEMMA. *Let G be a Gröbner basis in* $\mathbf{Q}[u, x]$.

The variables u are independent w.r.t. Ideal(G) iff $\neg(\exists g \in G)(g \in \mathbf{Q}[u])$.

THEOREM. *Let G be a Gröbner basis over* $\mathbf{Q}[u, x]$. *Then*

$$c \in Ideal_{\mathbf{Q}(u)[x]}(G) \text{ iff } ITPSRED(G, c, u) = (0, s).$$

EXAMPLE. For the above example, the Gröbner basis of the hypotheses over the rationals is:

$$\begin{aligned}
g_1 &= y_2^2 + 2y_1y_2 - 2y_1^2, & g_2 &= y_1^2y_3 - \tfrac{1}{2}y_1^3,\\
g_3 &= y_1y_2y_3 - \tfrac{1}{2}y_1^2y_2, & g_4 &= y_1y_4 - y_2y_3 - y_1y_3 + y_1y_2,\\
g_5 &= y_2y_4 + y_2y_3 + 2y_1y_3 - 2y_1y_2, & g_6 &= y_5 - \tfrac{1}{2}y_2,\\
g_7 &= y_6 - \tfrac{1}{2}y_2, & g_8 &= y_1^2y_7 - \tfrac{1}{2}y_1^3,\\
g_9 &= y_1y_2y_7 - \tfrac{1}{2}y_1^2y_2, & g_{10} &= y_1y_8 + y_2y_7 + y_1y_7 - y_1y_2,\\
g_{11} &= y_2y_8 - y_2y_7 - 2y_1y_7 + 2y_1y_2.
\end{aligned}$$

ITPSRED gives $r = 0$ for both c_1 and c_2, hence the theorem is confirmed.

5. Computing Time Statistics

Other investigators who proved geometry theorems, independently of us, using Buchberger's algorithm include S. C. Chou and W. F. Schelter of the University of Texas at Austin and D. Kapur of GE at Schenectady. Chou & Schelter (1986) described an algorithm that is essentially the same as our method RED. Kapur (1986) gives a refutational approach that completely solves problem P1 in an algebraically closed field. (But, as we saw, even if the answer to P1 over $\mathbb{C}$ is "no" the answer to the underlying geometrical question may still be yes.) In general, this method turns out to be much slower than our approaches. For a comparison of computing times see Kutzler & Stifter (1986*a*, *b*).

The two methods RED and PRED have been implemented by the authors based on an existing implementation of Buchberger's algorithm (Gebauer & Kredel, 1984) in the computer algebra system SAC-2/ALDES (Collins & Loos, 1976) on an IBM 4341.

In the statistics below we give the computing times in seconds for RED and PRED. We separate the times for the two main steps, i.e. the Gröbner basis computation and the (pseudo)reduction. An entry of 0·0 indicates that the computing time was below 0·05 seconds. For all examples we used the same working space of 600 000 cells (about 2·4 Mbyte). An entry "space" indicates that the available 2·4 Mbyte were not sufficient. Full descriptions of the examples can be found in the cited papers. The four columns giving the total degree of the hypotheses H and the conjecture c as well as the number of independent variables u and dependent variables x allow a classification of the examples. Further, the examples are ordered lexicographically on maximal degree of hypotheses/ number of variables/number of independent variables/number of dependent variables/ degree of conjecture. By the remarks we made on proper points, it seems pointless to list information about the polynomials s.

For examples with several conjectures, the first step of both methods need be done only once. In this case, the time-consuming computation of the Gröbner basis of the hypothesis polynomials can be regarded as "preprocessing". Furthermore, if all hypothesis polynomials are homogeneous (i.e. all terms have the same total degree) we may (and do) use a total degree ordering instead of a purely lexicographical ordering for method PRED. (As also observed in Buchberger (1985), a total degree ordering yields better computing times in many cases.)

Neither of the two methods is generally better than the other. PRED, in almost all examples, requires less computing time for the (pseudo)reduction step, whereas the Gröbner basis computation often takes a longer time. For the 70 examples contained in the statistics below the total time comparison yields the following result: For 32 examples RED is faster, for 31 examples PRED is faster, for 2 examples both methods require the same amount of time, and 5 examples could not be confirmed by either method due to working space problems.

Computing Time Statistics

		Degree		# var.		RED			PRED		
Theorem	Ref.	H	c	u	x	GB	NF	Σ	GB	IPR	Σ
Triangle_{51}	Ku/Sti 86*b*	1	2	3	5	0·7	0·3	**1·0**	1·0	0·0	**1·0**
Triang., ex. 3.5.3	Chou 85	1	2	3	5	0·1	0·3	**0·4**	0·9	0·1	**1·0**
Quadr. midpnts $\mathbb{R}^2$	Ku/Sti 86*b*	1	2	5	7	1·0	2·0	**3·0**	1·0	0·1	**1·1**
Iterated reflection	Ku/Sti 86*b*	1	1	5	10	0·2	0·0	**0·2**	1·2	0·0	**1·2**
Quadr. midpnts $\mathbb{R}^3$	Ku/Sti 86*b*	1	2	7	10	2·3	10·9	**13·2**	1·2	0·2	**1·4**
Square	Chou 84	1–2	2	1	4	1·2	0·0	**1·2**	1·2	0·0	**1·2**
Example	This paper	1–2	2	1	7	0·6	1·0	**1·6**	1·6	0·3	**1·9**
Trisector of side	Ku/Sti 86*b*	1–2	2	3	5	1·8	0·2	**2·0**	1·1	0·1	**1·2**
Angle bisection	Ku/Sti 86*b*	1–2	4	3	5	4·0	0·2	**4·2**	2·4	0·0	**2·4**
Triangle_{76}	Ku/Sti 86*b*	1–2	6	3	6	1·0	0·4	**1·4**	1·0	0·0	**1·0**
Pentag. gold. ratio	Ku/Sti 86*b*	1–2	5	1	9	5·3	3·2	**8·5**	8·7	1·2	**9·9**
9 point circle	Ch/Sch 86	1–2	2	3	7	10·4	0·3	**10·7**	28·0	0·2	**28·2**
Triangle centr. (1)	Ch/Sch 86	1–2	2	3	7	1·9	0·9	**2·8**	1·1	0·2	**1·3**
Triangle centr. (2)	Ku/Sti 86*b*	1–2	2	3	7	4·1	0·9	**5·0**	37·2	2·8	**40·0**
Circle equid. sec. (2)	Ku/Sti 86*b*	1–2	2	3	8	0·6	2·3	**2·9**	1·5	0·1	**1·6**
Gauss' T.	Ch/Sch 86	1–2	2	4	8	1·3	1·6	**2·9**	1·0	0·1	**1·1**
Orthic group II	Chou 85	1–2	2	5	14	4·8	10·3	**15·1**	1·5	0·4	**1·9**
Tetrahedron (1)	Ku/Sti 86*b*	1–2	2	6	14	20·5	6·3	**26·8**	6·8	1·0	**7·8**
Tetrahedron (2)	Ku/Sti 86*b*	1–2	2	1	22	7·1	1·4	**8·5**	190·3	0·5	**190·8**
Wang's T.	Ch/Sch 86	1–2	2	6	18	59·7	9·4	**69·1**	326·0	5·9	**331·9**
Invers._{83}	Ku/Sti 86*b*	2	2	2	3	0·3	0·2	**0·5**	0·9	0·0	**0·9**
Circle intersection	Ku/Sti 86*b*	2	2	2	3	0·9	0·0	**0·9**	1·0	0·0	**1·0**
Invers. angle bisect.	Ku/Sti 86*b*	2	4	3	2	0·1	0·6	**0·7**	0·9	0·0	**0·9**
Invers. tangent	Ku/Sti 86*b*	2	4	3	2	0·1	0·1	**0·2**	0·9	0·0	**0·9**
Triangle area	Chou 84	2	6	3	2	0·5	0·3	**0·8**	0·9	0·1	**1·0**

Table—*continued*

Theorem	Ref.	Degree		# var.		RED			PRED		
		H	*c*	*u*	*x*	GB	NF	Σ	GB	IPR	Σ
Regular 12-gon	Ku/Sti 86*b*	2	2	0	6	1·4	0·3	1·7	1·3	0·1	1·4
$Triangle_{50}$	Ku/Sti 86*b*	2	2	2	4	0·7	0·6	1·3	0·9	0·0	0·9
Triang., ex. 4.3.1	Chou 85	2	2	3	3	1·7	0·0	1·7	1·5	0·0	1·5
Parallelogram	Chou 84	2	2	3	4	2·0	0·1	2·1	1·1	0·0	1·1
Triangle Euler's line	Ku/Sti 86*b*	2	2	3	4	1·4	0·7	2·1	1·5	0·1	1·6
Harmonic points	Ku/Sti 86*b*	2	2	3	4	3·1	2·6	5·7	6·2	0·0	6·2
Triangle altitudes	Chou 84	2	4	3	4	2·1	2·4	4·5	1·9	0·0	1·9
Angle inversion	Ku/Sti 86*c*	2	4	3	4	0·6	0·0	0·6	1·0	0·0	1·0
Ptolemy's T.	Chou 85	2	8	4	3	0·0	1·5	1·5	0·8	3·9	4·7
Peripheral angle	Ku/Sti 86*b*	2	8	4	3	0·0	1·5	1·5	0·9	5·6	6·5
Orthic group I	Chou 85	2	2	5	2	0·1	0·0	0·1	0·9	0·0	0·9
Brahmagupta's T.	Ch/Sch 86	2	1	3	5	6·6	0·1	6·7	26·0	0·1	26·1
Isosceles midpoint	Ch/Sch 86	2	2	3	5	4·0	1·0	5·0	16·4	0·3	16·7
Triangle equal ang.	Ku/Sti 86*b*	2	2	4	4	2·8	9·9	12·7	2·2	0·2	2·4
Pentagon	Ku/Sti 86*b*	2	2	4	4	4·6	1·4	6·0	5·4	0·0	5·4
Desargues' T. (2)	Wu 82	2	2	5	3	3·5	0·2	3·7	1·2	0·0	1·2
Circle equal ang. (1)	Ku/Sti 86*b*	2	3	3	6	10·8	2·1	12·9	37·9	0·2	38·1
Circle secants	Ku/Sti 86*b*	2	4	4	5	52·9	199·4	252·3	43·5	54·9	98·4
Double proportion	Ku/Sti 86*b*	2	4	5	4	0·7	2·8	3·5	0·9	0·1	1·0
Feuerbach (simple)	Ku/Sti 86*a*	2	2	2	8	2·1	0·7	2·8	1·4	0·1	1·5
Ceva's T.	Ku/Sti 86*b*	2	6	5	5	6·2	11·1	17·3	5·6	0·2	5·8
Desargues' T. (1)	Ku/Sti 86*b*	2	2	6	4	6·0	0·3	6·3	1·2	0·1	1·3
Circle equid. sec. (1)	Ku/Sti 86*b*	2	2	3	8	30·8	4·0	34·8	6·1	0·7	6·8
Circle equal ang. (2)	Ku/Sti 86*b*	2	6	3	8	5·4	70·2	75·6	47·6	9·3	56·9
Simson's T. (2)	Chou 84	2	2	4	7	4·9	9·5	14·4	24·1	0·4	24·5
Trigonometry	Wu 78	2	3	2	10	1·1	0·2	1·3	1·2	0·0	1·2
Harmonic points (1)	Ku/Sti 86*c*	2	4	5	7	41·4	1096·3	1137·4	505·9	1272·2	1778·1
Pappus' T.	Chou 84	2	2	6	6	7·1	52·6	59·7	11·3	0·3	11·6
Triangle (AMM)	Ku/Sti 86*b*	2	2	3	10	1·2	0·2	1·4	1·5	0·2	1·7
Triang., ex. 5.2.2	Chou 85	2	4	3	10	7·4	1·9	9·3	25·6	2·1	27·7
Butterfly	Chou 84	2	1	4	9	3006·9	12·7	3019·6	space	—	space
Line inversion	Ku/Sti 86*b*	2	2	4	10	4·4	0·1	4·5	139·0	0·1	139·1
Pappus' dual	Ku/Sti 86*b*	2	2	7	7	173·1	308·6	481·7	334·4	0·2	334·6
Pascal's T. (circle)	Ch/Sch 86	2	2	6	10	1977·6	11357·6	13335·2	space	—	space
Pratt's T.	Chou 85	2	10	8	10	space	—	space	space	—	space
Pappus point	Ch/Sch 86	2	2	6	13	193·0	159·5	352·5	space	—	space
Pascal–Conic	Wu 84*b*	2	2	7	18	space	—	space	space	—	space
Altitudes concur.	Wu 82	3	5	3	5	2·6	0·4	3·0	4·0	4·0	4·0
Angle bisection	Ku/Sti 86*b*	2–4	2	3	5	40·1	3·0	43·1	54·4	2·3	56·7
Simson's T. (1)	Chou 84	2–4	2	4	5	4·7	9·7	14·4	44·8	0·3	45·1
Conic–Polare	Ku/Sti 86*b*	2–4	4	5	5	82·0	4927·8	5009·8	84·5	76·5	161·0
Harmonic points (2)	Ku/Sti 86*c*	2–4	2	5	6	348·9	1·5	350·4	space	—	space
Pascal's T. (ellips.)	Ku/Sti 86*b*	2–4	2	8	12	space	—	space	space	—	space
Pascal's dual (ellips.)	Ku/Sti 86*b*	2–4	2	8	20	space	—	space	space	—	space
Feuerbach/Inscribed	Wu 82	1–6	4	3	9	space	—	space	space	—	space

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References

Buchberger, B. (1965). *An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-dimensional Polynomial Ideal* (German). Univ. of Innsbruck, Inst. f. Mathematik, Ph.D. thesis.

Buchberger, B. (1970). An algorithmic criterion for the solvability of an algebraic system of equations (German). *Aequ. Math.* **4**, 374–383.
Buchberger, B. (1985). Gröbner bases—an algorithmic method in polynomial ideal theory. In (Bose, N. K., ed.) *Multidimensional Systems Theory.* Dordrecht: Reidel.
Chou, S. C. (1984). Proving elementary geometry theorems using Wu's algorithm. *Contemp. Math.* **29**, 243–286.
Chou, S. C. (1985). *Proving and Discovering Theorems in Elementary Geometries Using Wu's Method.* University of Texas, Ph.D. thesis.
Chou, S. C., Schelter, W. F. (1986). Proving geometry theorems with rewrite rules. Subm. to *J. Automated Reasoning.*
Collins, G. E. SAC-2 Manual. University of Wisconsin, Computer Science Department.
Gebauer, R., Kredel, H. (1984). Buchberger algorithm system. *SIGSAM Bull.* **18.**
Gröbner, W. (1949). *Modern Algebraic Geometry* (German). Berlin: Springer.
Kapur, D. (1986). Using Gröbner bases to reason about geometry problems. *J. Symb. Comp.* **2**, 399–408 (this issue).
Kutzler, B., Stifter, S. (1986*a*). Automated geometry theorem proving using Buchberger's algorithm. In: *Proc. SYMSAC '86*, Waterloo, Canada.
Kutzler, B., Stifter, S., (1986*b*). *Collection of Computerized Proofs of Geometry Theorems.* Univ. Linz, technical report.
Kutzler, B., Stifter, S. (1986*c*). *New Approaches to Computerized Proofs of Geometry Theorems.* Univ. Linz, preprint.
Loos, R., (1976). The algorithm description language ALDES. *SIGSAM Bull.* **10**, 14–38.
Tarski, A. (1948). *A Decision Method for Elementary Algebra and Geometry.* Univ. of Calif. Press.
Wu, W. T. (1978). On the decision problem and the mechanization of theorem proving in elementary geometry. *Scientia Sinica* **21**, 150–172.
Wu, W. T. (1982). Toward mechanization of geometry. Some comments on Hilbert's "Grundlagen der Geometrie". *Acta Math. Scientia* **2**, 125–138.
Wu, W. T. (1984*a*). Basic principles of mechanical theorem proving in elementary geometries. *J. Syst. Sci. Math. Sci.* **4**, 207–235.
Wu, W. T. (1984*b*). Some recent advances in mechanical theorem proving of geometries. *Contemp. Math.* **29**, 235–241.